Optimization of Low-Thrust Heliocentric Trajectories with Large Launch Asymptote Declinations

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The problem of optimizing electric propulsion heliocentric trajectories, including the effects of geocentric launch asymptote declination on launch vehicle performance capability, is formulated and a solution is developed using variational calculus techniques. The model of the launch vehicle performance includes a penalty associated with noneasterly launch plus another penalty arising from a noncoplanar launch from the parking orbit. Provisions for range safety constraints are included. The solution of the resultant optimization problem requires that the launch excess velocity be offset from the initial primer vector. Mission performance for example cases shown indicate a small to moderate performance degradation compared to corresponding cases where declination effects are ignored.

Introduction

PRELIMINARY performance studies of heliocentric electric propulsion missions require some means of correlating initial spacecraft mass m_0 and launch hyperbolic excess velocity V_∞. In most studies to date, this has been accomplished by equating m_0 to the launch vehicle (LV) payload m_ℓ, which is represented as a nonlinear function of the scalar quantity v_∞, the hyperbolic excess speed. With few exceptions, this LV payload capability assumed has been that corresponding to a due-East launch from the Eastern Test Range (ETR). The direction of the launch hyperbolic excess velocity is usually left unspecified, and is determined as part of the solution to the optimization problem. With the indirect optimization technique, the solution dictates that V_∞ be directed along the initial primer vector, a requirement that may be in conflict with the assumed LV payload capability.

If the point of the (coplanar) injection from a circular parking orbit is properly chosen, the geocentric declination δ of the hyperbolic excess velocity may lie within the range

$$-i \leq \delta \leq i \tag{1}$$

where i is the equatorial inclination of the parking orbit established by the launch vehicle. If the launch excess velocity asymptote declination, as determined by the solution to the optimization problem, falls within this range and if the LV payload capability is compatible with the orbit inclination i, then the solution is consistent within the assumptions made and the results are valid. However, if $|\delta| > i$, then the basic assumptions regarding LV capability are in conflict, and it is necessary to formulate the optimization problem to account for the depencence of LV payload on direction of the launch excess velocity asymptote.

Although the validity of published high asymptote declination solutions which neglect the influence of declination on LV capability has been questioned for some time, no formal treatment of the problem has been noted in the literature. The authors had previously developed[2] a technique for adjusting the LV payload to account for the noncoplanar injection maneuver required to achieve the geocentric declination of the primer vector, which was colinear with V_∞, but this a posteriori correction approach has proved unacceptable because the original transversality conditions were no longer valid. Typically, these conditions resulted in "solutions" which were not stationary points. This condition arose because the alignment of V_∞ with the initial primer was no longer a necessary condition of optimality, but rather an imposed constraint which was in violation of the assumptions used in originally formulating the solution.

In this paper, a unified treatment of the high asymptote declination problem is presented. The LV payload capability is modeled as a function not only of the magnitude of V_∞ but also of the inclination of the circular parking orbit and of the declination of the launch asymptote. The formulation permits the optimization of both the parking orbit inclination and asymptote declination or of the asymptote declination subject to a limitation on parking orbit inclination to satisfy range safety constraints. The necessary conditions of optimality are derived for a typical comet or asteroid rendezvous problem. Example problems which exhibit the high asymptote declination characteristic are then solved and the results are discussed.

Problem Formulation

High launch asymptote declinations frequently arise in missions to targets that have orbits highly inclined to the ecliptic, such as those to certain comets and asteroids. Therefore, we select, for illustrative purposes, an optimal rendezvous mission to a single, massless target whose path is defined by a specified ephemeris. The extension of the results derived here to other missions of interest, such as flybys, orbiters, and multiple-targer missions, is straightforward. We shall also assume a propulsion system of fixed size in terms of mass and reference power. Also, overall propulsion system efficiency and the specific impulse of the thruster subsystem are assumed given and are held constant throughout the mission.

The spacecraft is composed of several mass components as follows:

$$m_o = m_{ps} + m_p + m_t + m_s + m_{\text{net}} \tag{2}$$

where m_0 is the initial spacecraft mass, m_{ps} is the specified mass of the total propulsion system including arrays, thrusters and power conditioners, m_p is the propellant mass, m_t is the tankage mass, m_s is the structure mass, and m_{net} is the net spacecraft mass, which is to be maximized. The propellant mass is conveniently expressed in terms of the final

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mass ratio ν_f

$$m_p = m_0(1-\nu_f) \tag{3}$$

and the tankage and structure masses are assumed to be linear functions of the propellant and initial spacecraft masses, respectively

$$m_t = k_t m_p \tag{4a}$$

$$m_s = k_s m_0 \tag{4b}$$

where k_t and k_s are specified constants. The initial spacecraft mass is assumed to equal the launch vehicle payload capability, m_ℓ.

A solar electric propulsion system shall be assumed where the instantaneous power developed by the arrays varies with the distance from the sun according to a specified law, which we denote $\gamma(r)$, with r being the solar distance in AU. Mathematically, we write the instantaneous power p_i in terms of the reference power p_{ref} at 1 AU as

$$p_i = \gamma p_{\text{ref}}$$

Under these assumptions the instantaneous thrust acceleration a is written

$$a = 2\eta\gamma p_{\text{ref}}/\nu m_0 c = g\gamma/\nu \tag{5}$$

where η is the overall propulsion system efficiency, ν is the instantaneous mass ratio, c is the jet exhaust speed, and g represents a reference thrust acceleration which is equal to the thrust developed at 1 AU divided by the initial spacecraft mass.

The launch vehicle payload capability is assumed to follow the simple exponential law

$$m_\ell = b_1 e^{-(v_c/b_2)} - b_3 \tag{6}$$

where b_1, b_2, and b_3 are predetermined constants for each launch vehicle, and v_c is a characteristic speed representative of the energy required to achieve a specific escape trajectory. For example, for a due-East launch from ETR and a coplanar injection maneuver, v_c is defined to be the speed required at departure from a low-altitude circular reference orbit to achieve a specified hyperbolic excess speeed v_∞, i.e.,

$$v_c = (v_\infty^2 + 2v_0^2)^{1/2} \tag{7}$$

where v_0 is circular satellite speed in the reference orbit. Thus, for due-East launches and coplanar injection maneuvers, m_ℓ is a function only for v_∞ for a given launch vehicle. Performance data for a large selection of existing and potential launch vehicles are presented graphically in Ref. 1 as a function of v_c, as previously defined with the reference orbit altitude being 185 km. The authors have found a least-squares curve fit to the exponential law above using, say 7-10 data points from a given payload curve to be quite adequate and accurate representation of the performance capability of a launch vehicle.

To accommodate large launch asymptote declinations, the same exponential law for launch vehicle payload may be used, but the definition of characteristic speed must be expanded to reflect the additional energy required to rotate the asymptote. This new definition of v_c is taken to be that given above plus the velocity penalties associated with the asymptote rotation. The rotation is assumed to be accomplished by first choosing a launch azimuth which establishes a given reference orbit inclination i followed by a noncoplanar injection maneuver from that circular reference orbit to the desired asymptote. The velocity penalty incurred with non-due-East launches from the ETR is shown graphically in Ref. 1 as a function of the orbit inclination. This velocity penalty, which we will denote Δv_i, is adequately approximated with a quadratic curve fit of the form

$$\Delta v_i = c_1 i^2 + c_2 i + c_3 \tag{8}$$

Normal range safety limitations restrict the range of inclinations achievable through varying the launch azimuth alone. The referenced graph indicates that the maximum allowable northerly azimuth will yield an orbital inclination of about 48.5°, while the maximum allowable southerly azimuth will yield an inclination of about 32°. Now, given a reference orbit inclination i, it remains to define the velocity penalty Δv_g associated with a noncoplanar departure from this circular orbit to the desired hyperbolic excess velocity at a declination δ. Assuming the line of nodes of this reference orbit is an open variable, one may choose this variable to minimize the angle between the excess velocity and the orbital plane. This minimum angle is $\delta - i$. Gunther[3] has shown that the minimum incremental velocity required to achieve a given v_∞ along an asymptote not lying in the orbital plane from a specified circular orbit is obtained from the solution to a quartic equation in the sine of the out-of-plane angle. Defining

$$s = \sin(\delta - i) \quad \rho = v_\infty / v_0$$

$$p = s^2(\rho^2 + 4)$$

$$q = s^2(1-s^2)\rho^2 \tag{9a}$$

$$x = [((q/2)^2 + (p/3)^3)^{1/2} + q/2]^{1/3} - [((q/2)^2 + (p/3)^3)^{1/2} - q/2]^{1/3} \tag{9b}$$

$$y = [\rho^2/4 - x]^{1/2} \tag{9c}$$

$$w = \tfrac{1}{2}\{\rho/2 + y + [(\rho/2+y)^2 + 4(x/2 + (x^2/4 + s^2)^{1/2})]^{1/2}\} \tag{9d}$$

then Gunther's solution for the magnitude of the minimum velocity impulse required to accomplish the maneuver is

$$v_g = v_0[\rho^2 + 3 - 2((1 + \rho w - w^2)(2 + \rho w))^{1/2}]^{1/2} \tag{10}$$

and the penalty Δv_g is the difference between v_g and the velocity increment required if the out-of-plane angle is zero, i.e.,

$$\Delta v_g = v_g - [(v_\infty^2 + 2v_0^2)^{1/2} - v_0]$$

Thus, the definition of the characteristic speed for those cases in which the asymptote declination lies outside the interval given in Eq. (1) is

$$v_c = (v_\infty^2 + 2v_0^2)^{1/2} + \Delta v_i + \Delta v_g$$
$$= v_0 + v_g + \Delta v_i \tag{11}$$

We shall employ this definition in the formulation of the solution to the optimization problem.

State and Adjoint Equations

The second-order differential equation governing the motion of a low-thrust spacecraft in heliocentric space is

$$\ddot{R} = h_\sigma g\gamma \bar{e}_t/\nu - \mu R/r^3 \tag{12}$$

where R is the heliocentric position vector, r is the magnitude of R, μ is the gravitational constant of the sun, $\bar{e}_t$ is a unit vector in the direction of thrust, and h_σ is a step function equal to one when the low-thrust engines are operating, and equal to zero otherwise. Since the thrust direction, as a function of time, and the engine switching times are to be determined from the solution to the optimization problem, both $\bar{e}_t$ and h_σ are control variables. The mass ratio satisfies the first-order differential equation

$$\dot{\nu} = -h_\sigma g\gamma/c \tag{13}$$

while γ is determined from a given algebraic equation as a function of r, μ is a known constant, and g and c are constants over the mission. For the problem statement previously given, c is specified, and g is defined by Eq. (5) to yield the desired reference power. Since the value of g is unknown, it is convenient to treat g as a state variable satisfying the equation

$$\dot{g} = 0 \tag{14}$$

with the initial condition being determined as a portion of the solution to the optimization problem. The Eqs. (12-14) constitute the state equations for the problem under consideration.

Applying the standard rules of the calculus of variations to these state equations, we find the corresponding adjoint (or co-state) equations may be written by insepction as follows

$$\ddot{\Lambda} = h_\sigma g\gamma'(\Lambda\cdot\bar{e}_t - \nu\lambda_\nu/c)R/\nu r + 3\mu(\Lambda\cdot R)R/r^5 - \mu\Lambda/r^3$$

$$\dot{\lambda}_\nu = h_\sigma g\gamma(\Lambda\cdot\bar{e}_t)/\nu^2$$

$$\dot{\lambda}_g = -h_\sigma\gamma(\Lambda\cdot\bar{e}_t - \nu\lambda_\nu/c)/\nu$$

where $\gamma' = \partial\gamma/\partial r$, Λ is the familiar primer vector which is adjoint to the velocity, and λ_ν and λ_g are adjoint variables associated with the mass ratio and reference thrust acceleration, respectively.

Boundary Conditions

The numerical solution of the state and adjoint equations requires the specification of the initial values of each of the variables. Some of these initial conditions are completely specified by the problem statement, while others must be determined to satisfy desired final conditions. The initial position R_0 may be regarded as a function of launch date t_0 only, i.e.,

$$R_0 = P_\ell(t_0)$$

where P_ℓ is the position of the launch planet as defined by a prespecified ephemeris. The spacecraft initial velocity in heliocentric space is taken to be the vector sum of the velocity of the launch planet and the hyperbolic excess velocity V_∞ attained with the specified launch vehicle.

$$\dot{R}_0 = \dot{P}_\ell(t_0) + V_\infty$$

Both the magnitude and direction of V_∞ are unspecified and therefore must be determined in the solution of the optimization problem. The initial mass ratio is, by definition,

$$\nu_0 = 1$$

whereas $g(t_0) = g$ is obtained from Eq. (5); i.e.,

$$g = 2\eta p_{\text{ref}}/m_0 c \tag{15}$$

The initial primer Λ_0 and its time derivative $\dot{\Lambda}_0$ are both unknown but must be chosen so as to satisfy the desired final conditions in position and velocity which, for rendezvous problem, are

$$R_f = P_t(t_f)$$

$$\dot{R}_f = \dot{P}_t(t_f)$$

where P_t is the position vector of the target as defined by the specified target body ephemeris at the final time t_f. The initial values of λ_ν and λ_g are also unknown and must be selected to satisfy appropriate transversality conditions as described following.

Optimality Conditions

The necessary conditions of optimality for the problem posed are easily derived by applying the maximum principle to the variational Hamiltonian h_v, which is a constant of the motion and is defined

$$h_v = \Lambda\cdot\ddot{R} - \dot{\Lambda}\cdot\dot{R} + \lambda_\nu\dot{\nu} + \lambda_g\dot{g}$$

$$= h_\sigma g\gamma(\Lambda\cdot\bar{e}_t - \nu\lambda_\nu/c)/\nu - \mu(\Lambda\cdot R)/r^3 - \dot{\Lambda}\cdot\dot{R}$$

The proper choice for the value of h_σ is seen to depend on the sign of the switch function σ, where

$$\sigma = \Lambda\cdot\bar{e}_t - \nu\lambda_\nu/c$$

Since $g\gamma/\nu$ must be nonnegative to have any physical significance, h_v is maximized with respect to h_σ by choosing $h_\sigma = 0$ if $\sigma < 0$, $h_\sigma = 1$ if $\sigma > 0$, and with respect to $\bar{e}_t$ by choosing

$$\bar{e}_t = \Lambda/\lambda$$

where $\lambda = |\Lambda|$. These well-known results constitute the definition of the optimal control variables of the problem. We now define the conditions for selecting any open end conditions which, for the problem under consideration, are the launch excess velocity, the inclination of the launch parking orbit and, possibly, the launch and/or arrival dates. The conditions that are sought are termed transversality conditions and are obtained from the general equation

$$-dm_{\text{net}} + [\Lambda\cdot d\dot{R} - \dot{\Lambda}\cdot dR + \lambda_\nu d\nu + \lambda_g dg - h_v dt]_0^f = 0 \tag{16}$$

subject to all constraints imposed in the formulation.

Reorganizing Eq. (2) and substituting Eqs. (3) and (4) yields m_{net}

$$m_{\text{net}} = -m_0[k_s + k_t - (1+k_t)\nu_f] - m_{ps}$$

such that

$$-dm_{\text{net}} = [k_s + k_t - (1+k_t)\nu_f]dm_0 - m_0(1+k_t)d\nu_f$$

where, from Eq. (6)

$$dm_0 = dm_\ell = -(b_1/b_2)e^{-(v_c/b_2)}dv_c \tag{17}$$

Clearly, the definition of the total differential dv_c is the key to optimizing the launch conditions for high asymptote cases. For the familiar case in which v_c is defined by Eq. (7) and is a function only of v_∞, one obtains

$$dv_c = (v_\infty/v_c)dv_\infty$$

However, for the high declination case in which v_c is given by Eq. (11), the differential dv_c is

$$dv_c = (\partial v_g/\partial v_\infty)dv_\infty + (\partial v_g/\partial\delta)d\delta$$

$$+ (\partial v_g/\partial i + \partial\Delta v_i/\partial i)di$$

where, from Eq. (8)

$$\partial\Delta v_i/\partial i = 2c_1 i + c_2$$

and from Eqs. (9) and (10)

$$\partial v_g/\partial i = -\partial v_g/\partial\delta$$

The derivation of the partial derivatives $\partial v_g/\partial v_\infty$ and $\partial v_g/\partial\delta$ is straightforward, although somewhat cumbersome. The equations are presented in the Appendix.

From the stated boundary conditions, the following differentials are written by inspection:

$$dR_0 = \dot{P}_\ell dt_0$$

$$d\dot{R}_0 = \ddot{P}_\ell dt_0 + dV_\infty$$

$$d\nu_0 = 0,$$

$$dR_f = \dot{P}_t dt_f,$$

$$d\dot{R}_f = \ddot{P}_t dt_f,$$

and, since g is a constant

$$dg_0 = dg_f = (2b_1\eta p_{\text{ref}}/b_2 m_0^2 c)\, e^{-(v_c/b_2)} dv_c$$

To avoid unnecessary algebraic manipulations later, it is convenient at this point to replace the vector differential dV_∞ in favor of differentials of parameters already appearing in the problem, which include dv_∞ and $d\delta$. In general, any vector may be uniquely defined in terms of its magnitude and rotations about two arbitrary orthogonal unit vectors, say $\bar{a}$ and $\bar{b}$, provided the original vector is not contained in the plane of $\bar{a}$ and $\bar{b}$. Denoting as α and β the rotation angles about $\bar{a}$ and $\bar{b}$, respectively, then dV_∞ may be written

$$dV_\infty = (V_\infty/v_\infty)dv_\infty + (\bar{a}\times V_\infty)d\alpha + (\bar{b}\times V_\infty)d\beta$$

To simplify subsequent algebraic relations, one may choose $\bar{a}$ and $\bar{b}$ such that one of the angles, say β, is equal to δ. This is accomplished by choosing $\bar{a}$ in the direction of the earth's North Pole, i.e., along the vector $\bar{n}_p$, and defining

$$\bar{b} = (V_\infty \times \bar{n}_p)/|V_\infty\times\bar{n}_p| \tag{18}$$

Thus, α represents the right ascension of the departure asymptote and β is the declination. Substituting the differentials previously defined into Eq. (16) and collecting coefficients of all remaining differentials yields

$$\begin{aligned} &f(\partial\Delta v_i/\partial i - \partial v_g/\partial\delta)di \\ &\quad + [f(\partial v_g/\partial v_\infty) - (\Lambda_0\cdot V_\infty)/v_\infty]dv_\infty \\ &\quad + [f(\partial v_g/\partial\delta) - \Lambda_0\cdot(\bar{b}\times V_\infty)]d\delta - \Lambda_0\cdot(\bar{n}_p\times V_\infty)d\alpha \\ &\quad + (\Lambda_f\cdot\ddot{P}_t - \dot{\Lambda}_f\cdot\dot{P}_t - h_v)dt_f \\ &\quad - (\Lambda_0\cdot\ddot{P}_\ell - \dot{\Lambda}_0\cdot\dot{P}_\ell - h_v)dt_0 \\ &\quad + [\lambda_{\nu f} - m_0(1+k_t)]d\nu_f = 0 \end{aligned} \tag{19}$$

where

$$f = [k_s + k_t - (1+k_t)\nu_f - g(\lambda_{gf}-\lambda_{g0})/m_0](dm_\ell/dv_c) \tag{20}$$

with dm_ℓ/dv_c being the coefficient of dv_c in Eq. (17). Since each of the differentials in Eq. (19) is independent of the others, its coefficient must vanish, thereby yielding the remaining necessary conditions. Thus

For optimum launch parking orbit inclination

$$f(\partial\Delta v_i/\partial i - \partial v_g/\partial\delta) = 0 \tag{21}$$

For optimum launch excess speed

$$f(\partial v_g/\partial v_\infty) - (\Lambda_0\cdot V_\infty)/v_\infty = 0 \tag{22}$$

For optimum launch asymptote declination

$$f(\partial v_g/\partial\delta) - \Lambda_0\cdot(\bar{b}\times V_\infty) = 0 \tag{23}$$

For optimum launch asymptote right ascension

$$-\Lambda_0\cdot(\bar{n}_p\times V_\infty) = 0 \tag{24}$$

For optimum launch date

$$h_v + \dot{\Lambda}_0\cdot\dot{P}_\ell - \Lambda_0\cdot\ddot{P}_\ell = 0 \tag{25}$$

For optimum arrival date

$$\Lambda_f\cdot\ddot{P}_t - \dot{\Lambda}_f\cdot\dot{P}_t - h_v = 0 \tag{26}$$

For optimum final mass ratio

$$\lambda_{\nu f} - m_0(1+k_t) = 0 \tag{27}$$

If in any specific problem an end condition which previously was assumed open is, in fact, specified, then its differential is zero and its coefficient need not vanish. For example, if the inclination of the parking orbit resulting from the satisfaction of Eq. (21) above exceeds a range safety limit, then it is necessary to fix i at that limit and ignore the transversality condition, Eq. (21). Also, if the launch date and/or arrival date is specified, then the corresponding transversality conditions, Eqs. (25) or (26), respectively, are ignored. Or if both launch and arrival dates are unspecified but flight time is fixed, then $dt_f = dt_0$ and Eqs. (25) and (26) are replaced in favor of one condition represented by their sum; i.e.

$$\Lambda_f\cdot\ddot{P}_t - \dot{\Lambda}_f\cdot\dot{P}_t - \Lambda_0\cdot\ddot{P}_\ell + \dot{\Lambda}_0\cdot\dot{P}_\ell = 0 \tag{28}$$

A number of observations may be made at this point, which facilitate the understanding and implementation of the transversality equations. First, one will note that the variable λ_g never appears on the right-hand side of the state or adjoint equations and is not required in determining the optimal control. The only place λ_g appears is in Eq. (20) for f, and there only in the form $(\lambda_{gf} - \lambda_{g0})$. Consequently, the choice of λ_{g0} is completely arbitrary, having no bearing on the solution. Thus, for convenience, one may use as an initial condition

$$\lambda_{g0} = 0$$

Secondly, Eq. (24) implies that the right ascension of V_∞ must be equal to, or 180° from, that of Λ_0; i.e., V_∞ must lie in the plane of Λ_0 and $\bar{n}_p$. Then also $\bar{b}\times V_\infty$, where $\bar{b}$ is defined by Eq. (18), must also lie in this plane. If the first term in Eq. (23) were zero, which is the result obtained when the effects of declination are ignored in the formulation, then one obtains from Eqs. (23) and (24) the familiar result that V_∞ must be colinear with Λ_0. Usually, it is assumed that V_∞ is aligned with Λ_0, however cases have been found[4] for which the optimum solution resulted in V_∞ being diametrically opposed to Λ_0. The fact that the first term in Eq. (23) is nonzero means that V_∞ will be offset from Λ_0 by a finite angle. This offset, as previously noted, must be in the plane of Λ_0 and $\bar{n}_p$ and, intuitively, we know it must be in the direction of the equator so as to reduce v_c. The amount of the offset of V_∞ from Λ_0 may not be determined from Eq. (24) as an initial value problem since the variable f is a function of variables (ν and λ_g) evaluated at the final time. Thus, δ must be treated as an additional independent parameter, and Eq. (23) becomes another condition to be satisfied in the boundary value problem.

Finally, the satisfaction of Eq. (21) requires that the term within parentheses vanishes since f will normally be a nonzero quantity. Therefore, since the two partial derivatives are functions only of initial conditions, one may solve for the i that causes the parenthesized term to vanish and thereby eliminate the condition, Eq. (21), from the boundary value problem. Due to the complexity of the equations defining $\partial v_g/\partial\delta$, this solution for i must be obtained using an iterative technique.

The approach to the solution of the problem as previously formulated differs in three basic respects from that of the problem where asymptote declination is ignored: 1) the condition, Eq. (21), must be solved for the optimum parking orbit inclination, given values of v_∞ and δ; 2) the asymptote declination δ must be introduced as an independent parameter and Eq. (23) added as an end condition of the problem; and 3) the evaluation of V_∞ becomes somewhat more involved. The computation of V_∞, given v_∞, Λ_0, and δ, proceeds as follows. Denote as ϵ the obliquity of the ecliptic such that the matrix

$$\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon \\ 0 & \sin\epsilon & \cos\epsilon \end{bmatrix}$$

operating on a vector expressed in ecliptic Cartesian coordinates yields the same vector in earth equatorial coordinates. Then the right ascension α_λ of the initial primer Λ_0 may be written

$$\alpha_\lambda = \tan^{-1}[(\lambda_{y0}\cos\epsilon - \lambda_{z0}\sin\epsilon)/\lambda_{x0}]$$

where λ_{x0}, λ_{y0}, λ_{z0} are the given ecliptic coordinates of Λ_0. Then, the right ascension of the asymptote is set

$$\alpha = \alpha_\lambda \text{ or } \alpha = \alpha_\lambda + \pi$$

and V_∞ is evaluated

$$V_\infty = v_\infty \Phi^T \begin{bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{bmatrix}$$

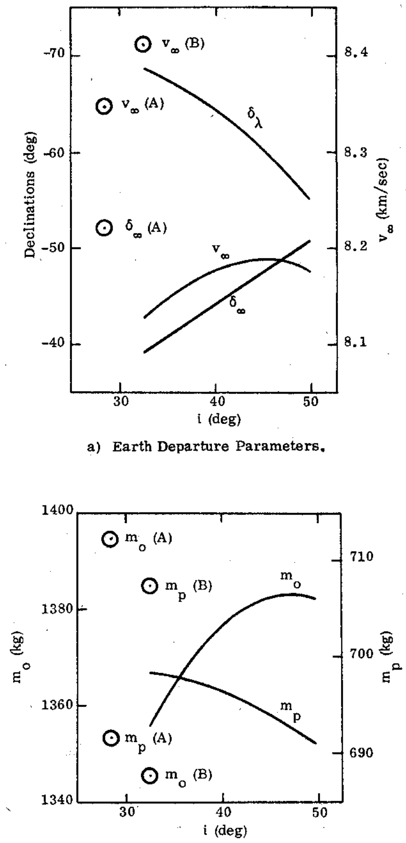
This may be contrasted with the usual definition of V_∞

$$V_\infty = v_\infty \Lambda_0 / |\Lambda_0|$$

Sample Problems

The solution described in the preceding paragraph was implemented in the variational calculus computer program HILTOP, and several specific trajectory studies have been performed, employing the Titan III E/Centaur/Sert 3 spacecraft launched from the ETR. The basic results of two of these studies are presented below. The first mission considered is the 45° extra-ecliptic mission terminating in a 1 AU circular orbit, and the second is a multiple target mission, launched in March 1985, flying past the two asteroids Metis and Amherstia, and terminating at rendezvous with the Comet Encke 30 days before perihelion in 1987. The results achieved for these missions are felt to be typical of those requiring nominally high launch asymptote declinations. These results are summarized in Figs. 1 and 2 as functions of the circular parking orbit inclination i.

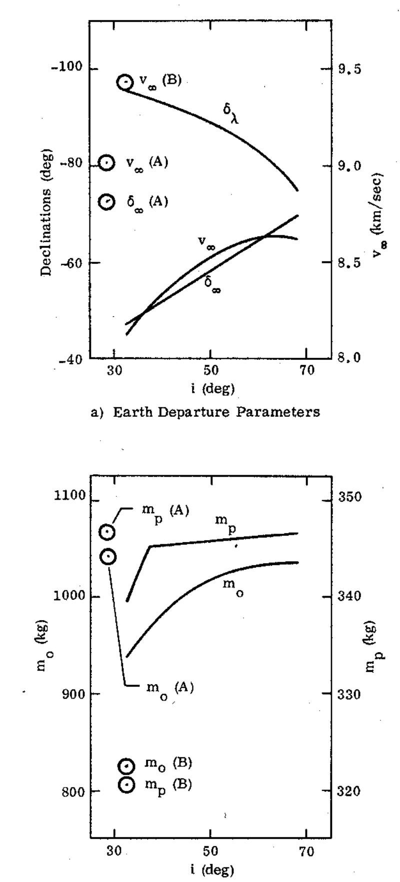
In these figures, δ_∞ is the launch asymptote declination (of V_∞) and δ_λ is the declination of the initial primer vector, Λ_0, along which the thrust vector of the low thrust spacecraft is initially oriented. These parameters, along with the excess speed, v_∞, the initial mass injected into heliocentric space, m_0, and the low-thrust propellant mass, m_p, are plotted as functions of parking orbit inclination from $i = 32.5°$ up to the value of i at which the final mass $m_0 - m_p$ reaches a maximum. Since the assumed range safety limit of the Titan III E/Centaur is $i = 32.5°$, all of the data in the plotted curves except the leftmost points (at $i = 32.5°$) are actually not allowable for that launch vehicle, but could be achieved by



a) Earth Departure Parameters.

b) Initial and Propellant Masses.

Fig. 1 Extra-ecliptic mission.



a) Earth Departure Parameters

b) Initial and Propellant Masses

Fig. 2 Multiple target mission.

that vehicle if the range safety constraint were relaxed. The plotted curves may, therefore, be considered to represent the general behavior of an arbitrary launch vehicle.

Some general trends are apparent upon inspection of the figures. As inclination increases, the excess speed v_∞, which is optimized, tends to increase along a curve similar to m_0, which is a behavior opposite to the simplistic dependence of m_0 on v_∞ usually assumed. m_0 increases in the face of increasing v_∞ because the noncoplanar burn angle $|\delta_\infty| - i$ out of the parking orbit decreases with increasing i, becoming very small where m_0 peaks. The offset angle of the initial low thrust vector, $|\delta_\lambda - \delta_\infty|$, also decreases with increasing i. The low-thrust propellant variation is very small as i varies. (The corner in the m_p curve of Fig. 2 is due to the introduction of a coast phase in the trajectory at lower values of i).

Points in the figures marked (A) were generated under the assumption of ignoring the departure asymptote declination, as has been done in most published solutions to date. These points therefore, represent erroneously optimistic data.

Points marked (B) were generated under the assumption of a coplanar injection out of the parking orbit, such that the launch vehicle is forced to launch into a parking orbit having an inclination equal to the departure asymptote declination. In the figures, these points correspond to $|\delta_\infty| = i = 32.5°$, the assumed range safety limit of the Titan III E/Centaur. These points, therefore, represent missions in which the parking orbit inclination is not optimized, and the low-thrust spacecraft must provide additional plane-change energy to compensate for the constrained launch vehicle performance, even though the launch vehicle provides more v_∞ to the low-thrust spacecraft. This represents a valid option in terms of performing the launch phase of a mission, and the penalty incurred appears to be small unless the departure asymptote declination is exceedingly large.

For the extra-ecliptic mission in Fig. 1, the optimal asymptote declination at point (A) is about $-52°$ and the final spacecraft mass is 703 kg. This will be termed the reference case in the comparisons following. Upon including the declination effects in the launch vehicle model, the optimum parking orbit inclination becomes 49.7° and the asymptote declination drops to $-50.6°$. This implies that the final launch injection maneuver is only 0.9° out of the parking orbit plane and illustrates the fact that high declinations are achieved at less cost with inclination changes than with out-of-plane injections. The declination of the initial primer vector is $-55°$, representing an offset angle of the hyperbolic asymptote of about 4.6°. The final spacecraft mass for this case is 691 kg, representing a loss of about 1.7%, compared to the reference case. Constraining the parking orbit inclination to the assumed range safety limit of 32.5° results in somewhat more severe penalties. The asymptote declination for this limiting inclination is $-39.2°$, representing an out-of-plane injection angle of about 6.7°. The declination of the initial primer is $-68.5°$, which corresponds to an asymptote offset angle of 29.3°. The final mass is 657 kg, representing a loss of 6.5% compared to the reference case. Finally, for the case in which the asymptote is forced to lie in the 32.5° inclined parking orbit plane, the asymptote offset angle jumps to $-39°$ and the final mass drops to 638 kg, a loss of 9.2%, compared to the reference case.

Considerably more substantial penalties are observed for the multiple target mission, principally because the nominal asymptote declination of $-72.7°$ for the reference case [i.e point (A)] represents a more severe problem to overcome. The final mass for the reference case is 695 kg. When the asymptote declination effects are introduced into the solution, the parking orbit inclination optimizes to a value of 68.2°, the asymptote declination becomes $-69.5°$, and the primer declination is $-74.8°$. Again the out-of-plane injection angle and the asymptote offset angle are quite small for this fully optimized case. The final mass is off less than 1% to 688.5 kg. Restricting the parking orbit inclination to 32.5° leads to an asymptote declination of $-46.8°$, corresponding to an out-of-plane injection angle of 14.3°, and to a primer vector declination of $-95.6°$, representing an asymptote off-set angle of 48.8°. Note that the primer vector has actually swept through the South Pole (corresponding to $\delta_\lambda = -90°$), and, therefore, the right ascension of the primer and excess velocity vectors differ by π rad. The final mass drops nearly 100 kg to 597 kg, a loss of about 14%. By further restricting the solution to force the launch asymptote to lie in the 32.5° inclined parking orbit plane, an additional significant penalty is incurred resulting in an overall loss of final mass of 27.6%. The final mass for this case is 503 kg and the asymptote offset angle is 85.4°.

Conclusions

From the two cases discussed, it appears that the accomodation of trajectories with high asymptote declinations is possible with a very minor performance penalty if range safety constraints are not imposed. In such a case, one simply accepts the relatively small launch vehicle azimuth penalty and establishes a parking orbit inclination nearly equal to the original asymptote declination. From this orbit, an injection maneuver with a small out-of-plane angle leads to a solution representing a performance loss of less than 1% for both missions investigated. The imposition of range safety constraints results in much more significant performance penalties. Limiting the parking orbit inclination to 32.5° resulted in a performance penalty of 6.5% for the extra-ecliptic mission where the original asymptote declination was $-52°$, and a penalty of 14% for the multiple target mission with an original declination of $-72.7°$. Further constraining the solution to force the asymptote to lie in the parking orbit plane increases the performance penalty by an amount ranging from negligible to significant, depending on the severity of the original declination problem. For the multiple target mission, this latter constraint alone accounted for a 13.6% performance degradation.

Finally, it should be noted that the analysis developed here is general and may be applied to any launch vehicle, launch site, or set of range safety constraints. It may also be applied to purely ballistic or high thrust heliocentric trajectories, although the option of forced coplanar launch will not be generally available if there exists no other propulsive means of subsequently changing the trajectory to target the spacecraft. The numerical approach outlined herein has been successfully implemented in a variational calculus computer program, and the ability to obtain optimal and suboptimal solutions efficiently under a variety of constraints has been proved.

Appendix

The equations for the partial derivatives $\partial v_g / \partial v_\infty$ and $\partial v_g / \partial \delta$ are derived from Eqs. (9) and (10) and are listed following:

$$\frac{\partial v_g}{\partial v_\infty} = \frac{v_0^2}{v_g} \{ \rho \frac{\partial \rho}{\partial v_\infty} - \frac{w(3+2\rho w - w^2)(\partial \rho / \partial v_\infty)}{2[(1+\rho w - w^2)(2+\rho w)]^{1/2}}$$

$$+ \frac{(3\rho + 2\rho^2 w - 3\rho w^2 - 4w)(\partial w / \partial v_\infty)}{2[(1+\rho w - w^2)(2+\rho w)]^{1/2}} \}$$

$$\frac{\partial v_g}{\partial \delta} = - \frac{v_0^2 (3\rho + 2\rho^2 w - 3\rho w^2 - 4w)}{2 v_g [(1+\rho w - w^2)(2+\rho w)]^{1/2}} \frac{\partial w}{\partial \delta}$$

where

$$\partial \rho / \partial v_\infty = 1/v_0$$

$$\frac{\partial w}{\partial v_\infty} = ½\{ ½ \frac{\partial \rho}{\partial v_\infty} + \frac{\partial y}{\partial v_\infty}$$

$$+ [(1+ \frac{x}{2(x^2/4 + s^2)^{1/2}}) \frac{\partial x}{\partial v_\infty}$$

$$+\left(\frac{\rho}{2}+y\right)\left(\tfrac{1}{2}\frac{\partial\rho}{\partial v_\infty}+\frac{\partial y}{\partial v_\infty}\right)\Big]/(2w-\rho/2-y)\Big\}$$

$$\frac{\partial w}{\partial\delta}=\tfrac{1}{2}\Big\{\frac{\partial y}{\partial\delta}+\Big[\frac{\partial x}{\partial\delta}+\frac{x(\partial x/\partial\delta)+4s(\partial s/\partial\delta)}{2(x^2/4+s^2)^{1/2}}$$

$$+\left(\frac{\rho}{2}+y\right)\frac{\partial y}{\partial\delta}\Big]/(2w-\rho/2-y)\Big\}$$

$$\partial s/\partial\delta=\cos(\delta-i)$$

$$\partial y/\partial v_\infty=[(\rho/2)\partial\rho/\partial v_\infty-\partial x/\partial v_\infty]/2y$$

$$\partial y/\partial\delta=-(\partial x/\partial\delta)/2y$$

$$\frac{\partial x}{\partial u}=\frac{1}{6}\Big[\frac{(q/2)(\partial q/\partial u)+(p/3)^2(\partial p/\partial u)}{((q/2)^2+(p/3)^3)^{1/2}}$$

$$+\frac{\partial q}{\partial u}\Big]\,[((q/2)^2+(p/3)^3)^{1/2}+q/2]^{-2/3}$$

$$-\frac{1}{6}\Big[\frac{(q/2)(\partial q/\partial u)+(p/3)^2(\partial p/\partial u)}{((q/2)^2+(p/3)^3)^{1/2}}$$

$$-\frac{\partial q}{\partial u}\Big]\,[((q/2)^2+(p/3)^3)^{1/2}-q/2]^{-2/3}$$

with

$$u=v_\infty \text{ or } \delta,$$

$$\partial q/\partial v_\infty=2\rho s^2(1-s^2)(\partial\rho/\partial v_\infty)$$

$$\partial q/\partial\delta=2\rho^2 s(1-2s^2)(\partial s/\partial\delta)$$

$$\partial p/\partial v_\infty=2\rho s^2(\partial\rho/\partial v_\infty)$$

$$\partial p/\partial\delta=2s(\rho^2+4)(\partial s/\partial\delta)$$

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